# Approximation of Sensitivity Coefficients for Inverse Reservoir Modeling through Evaluation of Physical Flow Criteria

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*Production data integration is a difficult inverse problem. Optimization algorithms that optimize the parameter models to match available flow data, and then run the forward problem to assess the improvement are almost always employed. These algorithms are run iteratively and converge if the optimization is changing the parameters in the right direction. The optimization of the parameter models requires calculation of sensitivity coefficients that specify how each input parameter affects the flow response. Analytical calculation of sensitivity coefficients requires extensive modification to the flow simulation code. We propose an approximation of the sensitivity coefficients based on a physical understanding of the flow process. This would permits optimization without tinkering with the flow simulation code.*

*An understanding of the physical controls on the magnitude and sign of sensitivity coefficients used in reservoir model calibration provides the modeler with significant insight into the calibration process. Inverse problems are notoriously difficult; obtaining satisfactory solutions to inverse problems requires consideration of these physical constraints. We identify a relatively small number of criteria that can predict the sensitivity of pressure to changes in the permeability field (1) cells common to streamlines that include the response location under assessment will be the most sensitive cells within the model for that location, (2) regions of high permeability are sensitive if they are connected to a source or sink, (3) low permeability regions along a streamline will be sensitive because they form baffles to flow, and (4) the sign of the sensitivity coefficient is related to whether a cell location is up- or downgradient from the observation location within a streamtube. Cells that are downgradient will always have a negative sign. These criteria can be applied to obtain an approximation of the sensitivity matrix with a simple streamline based flow simulation. Inverse techniques such as SSC could then be applied.*

## Introduction

Calibration of flow models to production data requires an assessment of the relative sensitivity of each of the model parameters to the flow response. Once the model grid is constructed and appropriate boundary conditions are assigned, the population of the grid with heterogeneous parameter fields is the biggest problem. The flow response we normally consider is pressure. The spatially varying parameter we normally consider is permeability. An assessment of the sensitivity of model response with respect to adjustments of the model parameters is required in order to make educated updates to the model for an efficient calibration.

There are many approaches to model calibration, or parameter inversion, where the objective is to infer appropriate model parameters from a partially known system response. In this paper, the parameters refer to the assigned grid block permeability values, and the system response is known from pressure measurements. Many automated inversion procedures recast the inverse problem as an optimization problem in which an optimal set of parameter adjustments is found by minimizing an objective function. The objective function is generally a direct measure of the state of calibration of the model, for example, the squared difference between pressure data and model-calculated pressures for a set of observation wells. Gradient methods are a widespread class of automated calibration techniques in which the optimal parameter perturbation vectors (permeability field changes) are obtained by a gradient-based search, in which the gradient of the objective function with respect to the parameter perturbations determines the search direction. There are a number of reviews that the readers may refer to for more details (Yeh, 1986; Carrera et al., 1990; Chu et al., 1995). All gradient-based methods require the computation of *sensitivity coefficients*, which are the partial derivatives of model response at a particular location with respect to a parameter changes at all other locations in the domain.

The focus of this research was to analyze the structure of sensitivity coefficients. Sensitivity of a fluid flow model to the permeability field is a physical problem, governed by the physics of flow. Despite an extensive body of literature on obtaining sensitivity coefficients as a key aspect to solving inverse flow problems, little attention has been paid to their structure in relation to the permeability field and the flow model.

*Sensitivity coefficients*:

Sensitivity coefficients are the partial derivatives of model response at a location, *j* with respect to a parameter change at location, *i*. In this paper, we consider the sensitivity of hydraulic head at an observation well *j* to a change in the logarithm of permeability at a model cell *i*. The complete matrix of sensitivities may be expressed as:

$$S = \{s_{i,j}\} = \frac{\partial h_j(t)}{\partial \ln(k)_i}, \qquad i = 1,...N; j = 1,...nobs \tag{1}$$

where $h$ is hydraulic head in metres (m), $k$ is absolute permeability ($m^2$), *nobs* is the number of observation locations, and $N$ is the number of model grid blocks.

There have been a number of methods proposed for the calculation of the sensitivity matrix, otherwise called the Jacobian matrix (Sykes et al., 1985; Yeh, 1986; Chu et al., 1995). All of them are based on analytical expressions. Numerical implementation has always been the limiting factor in the practicality of automated calibration algorithms (Carrera et al., 1990).

At the heart of the calibration problem is the need to predict or identify areas of high sensitivity to avoid adjusting parameters that are of little to no consequence to the model calibration, thus minimizing the time spent in obtaining a solution to the inverse problem. The biggest advantage to using automatic calibration procedures over simple trial-and-error approaches is obviously the reduction in time spent. A disadvantage that prevents their widespread use in practice is that these methods are most often black boxes to the user and require highly-specialized code. As a result, the modeler often gains little to no insight from the calibration procedure. The modeler would hope to be led to recognize sensitive areas, problem areas and problem data. A more physically-intuitive approach is desired. Insight is gained when the modeler can identify the sensitive parameters in the model and understand why they are sensitive. The link between all calibration procedures is a quantification of the relative sensitivity of model response at calibration points to changes in the parameters at every area in the model domain.

Sequential Self Calibration (SSC) is an inverse technique that combines geostatistics and constrained linear optimization using a gradient-based search (Gomez-Hernandez et al., 1997). The method of computing the sensitivity coefficients is based on a classical finite-differenced formulation of the sensitivity equations (Yeh, 1986; Wen et al., 1997). The computational cost is high to compute the full matrix of sensitivity coefficients given a small number of observation wells and a modest number of grid cells. In order to obtain the full $N$ x *nobs* sensitivity matrix, we need to solve $N$ systems of $N$ linear equations. In practice, the parameter space is reduced by the concept of master points, which is a unique feature of SSC, such that only $m$ systems of $N$ equations is solved, where $m$ is the number of master points (Gomez-Hernandez et al., 1997; Wen et al., 1999). Since these sensitivity coefficients are required by most calibration methods, the immediate problem we are addressing in this short note is whether, through understanding the physical rules governing the sensitivity of model response with respect to the specific structure of the permeability field, we can come up with a fast and adequate approximation of the sensitivity matrix for general purpose flow model calibration problems. Specifically: *Given a permeability field, and a fully-defined flow model is there a set of criteria we can apply to outline the areas that most strongly impact the model response at a specified location, and obtain enough information to come up with a numerical approximation of the corresponding vector of sensitivity coefficients?*

## Method of Analysis and Results

The first step was to examine the solution of the sensitivity equations by SSC on a reference permeability field. A two-dimensional 1x1 km square domain was discretized as a 50 x 50 grid using a uniform cell size of 20 m x 20 m and a uniform thickness of 100 m. A production well was located at the center of the domain, plus 4 observation wells forming a regular five-spot pattern, with the observation wells spaced at a distance of approximately 280 m from the pumping well (Figure 1). The permeability structure was generated by an unconditional sequential Gaussian simulation using an anisotropic variogram with maximum continuity at an azimuth of 45 degrees (Figure 1). The reference permeability histogram is lognormally distributed with a ln(k) mean and variance of 6.0 and 3.0, respectively. The flow equations were solved in steady state to obtain an equilibrium production drawdown. Drawdown did not reach the boundaries of the domain, as they were constrained with a uniform hydraulic head of 100 m (Figure 1).

*Characteristics of the analytically-derived sensitivities*

The sensitivity of head at each of the 4 non-pumping observation wells with respect to permeability change at every model cell [i.e., Equation (1)] was obtained by solving the full set of sensitivity equations using an implementation of the SSC algorithm. The results are shown as four maps (Figure 2) representing the four sensitivity vectors corresponding to the positions of the observation wells. The following observations were noted:

a. There are typically a few regions of high sensitivity within vast regions of low sensitivity.

b. There is a systematic pattern to the sign of the sensitivity coefficients which is related to whether a cell location is up- or downgradient from the observation location.

c. There is a visual correlation between sensitive areas and regions of higher ln(k) (Figure 3). Upon closer examination, such regions of high k cells are only sensitive if they are connected to a source or sink in the domain (i.e., a specified head or flux boundary or a well) (Figure 2; bodies 1-3).

d. Cells of lower ln(k) may also be sensitive, specifically, where they comprise regions that interrupt an otherwise high k channel *along a flow path* (Figure 4). In other words, if we consider the permeability of grid blocks within an arbitrary streamline, if the configuration is not described by the previous point (c), the sensitive regions will coincide with cells of relatively lower permeability that form “baffles” to flow within that streamtube (Figure 2; bodies 4-9) .

e. Cells located along streamlines coincident with the observation location of interest will be sensitive

These observations led to the formulation of a simple rules-based algorithm called “Appsen” to predict the distribution of sensitivities once the flow solution has been obtained at any step during the calibration process, without the need to solve for them analytically. The rules that designate a cell as potentially sensitive are as follows:

**Rule 1**: Cells considered as potentially sensitive with respect to a given (observation) location occur within a streamline that contains that observation.

**Rule 2**: Regions of high k cells comprise sensitive areas if they form a continuous, relatively high-transmissivity flowpath between the location of interest and a source or sink in the domain.

**Rule 3**: Baffles formed by relatively low permeability cells within a given streamline will comprise the most sensitive cells for that streamtube.

**Rule 4**: If the cell is downgradient of the observation, its sensitivity has a negative sign.

**Rule 5**: If the cell is in a location beyond the radius of influence of the well, it has zero sensitivity.

These 5 rules are simplifications; they may be tuned to account for problem-specific features or other observations. Some general characteristics observed in Figure 2 will be mentioned in the next section.

*The Appsen algorithm*

The ln(k) distribution is separated into high and low values according to the permeability histogram. The idea is to re-classify the possibly continuous permeability histogram in terms of a binary variable. This is to enable an approximate establishment of criteria (c) and (d), corresponding to Rules 2 and 3 above. For this example, in which the permeability is lognormally distributed, we define a global cutoff corresponding to the median ln(k) such that cells exceeding the median are high k, and those below the median are classified as low k.

The connected bodies or regions of high-k cells are calculated with, for example, the `geo_obj` code available in the CCG Software Catalogue. The connected regions that are in contact with a source or sink are flagged. Cells within these regions will potentially be assigned a non-zero sensitivity according to Rule 2.

The next step is to solve for pressure and track particles forward and backward from observation locations. A particle source template is defined around the observation location defining an arbitrary width of the streamtubes to be considered for the problem (Figure 5). Cells intersected by a pathline are assigned a sensitivity indicator if they meet the criteria of Rule 2 or 3. By this rationale, only cells within a streamtube containing the observation are potentially sensitive for that observation location (Figure 5b).

Two minor inconsistencies with this approach are apparent upon examination of the analytically-calculated sensitivities. The first can be seen with respect to the sensitivities for observation 1 (Figure 2a). A streamtube centered on the flowpath drawn through observation 1 is almost entirely comprised of high ln(k) cells (Figure 3). The only sensitive cells along the flowpath between the observation location and the pumping well are those of relatively lower ln(k) (i.e., the baffles) (Figure 4). The Appsen approach will flag all the cells within the streamtube as sensitive because they meet the criteria of Rule 2. What is observed within the terminal half of the streamtube, however, is behaviour related to Rule 3, but to account for the case that low k is higher than the global cutoff, we would need a streamtube-specific binning of the ln(k) distribution. Secondly, we have stipulated that in order for a cell to be considered sensitive for a given observation location, the cell has to be within a streamtube containing that observation well. Note however, the stringers of sensitive cells that are outside the streamtube enveloping a given observation location (compare streamtubes in Figure 5b with the sensitivities of any corresponding observation in Figure 2). This is somewhat of an artifact, albeit a physically-correct one. Most of these stringers correspond to high-k bodies connected to the boundaries or the well. Consequently, the value of ln(k) at a cell within these high-flow channels is a potentially sensitive parameter to every location in the domain. This is a higher-order effect that we are neglecting with the proposed methodology. Any of these higher-order characteristics may be accounted for by tuning the rules-based approach outlined in this short note.

This simple rule-based algorithm results in a sparse matrix of zeros and 1's, where the non-zero elements are the sensitive cells with the appropriate sign for the sensitivity coefficient assigned. The most important aspect of coming up with approximate sensitivities is the correct sign. An exercise conducted in an earlier study showed that adding up to 50% noise to the analytically-derived sensitivities from SSC did not affect convergence of the parameter optimization algorithm which relies on the sensitivity matrix, whereas incorrect signs had a more significant effect. Nevertheless, an appropriate magnitude should be given to the sensitivity coefficients based on the spatial scale of the model and flow rates. The methodology for coming up with these magnitudes is a subject of further research; however, the following section outlines some considerations.

*Comparison of analytically-derived sensitivities and Appsen sensitivities*

A visual comparison of the cells flagged with sensitivity indicators via the Appsen approach (Figure 6) and the analytically-derived coefficients (Figure 2) shows that the method is a rough approximation. Figure 6 shows the cells that have been flagged with a binary indicator of sensitivity with the appropriate sign.

The proper way to compare the coefficients obtained from the two methods is via the information they provide to parameter inversion/calibration. SSC uses a linear approximation of pressure around proposed sets of parameter perturbations to avoid running the full forward flow model for every iteration of the optimization routine. The optimization routine converges to an optimal set of permeability changes which minimize the mismatch between measured pressures and model-calculated pressures. After the optimization converges, the full flow solution is obtained to re-assess the value of the objective function, since there are some errors incurred through the linear approximation of pressure.

The linear approximation of the head at an observation location as a result of perturbing the permeability field can be expressed as

$$h_j^{(lin)} = h_j^{\,0} + \sum_{i=1}^{N}\left[\frac{\partial h_j}{\partial \Delta \ln(k)_i} \cdot \Delta \ln(k)_i\right], \qquad (2)$$

where: $h_j^{(lin)}$ is the linear approximation of head around a ln(k) perturbation field, $h_j^0$ is the base case head (the observation datum) before the proposed ln(k) perturbations, $\Delta\ln(k)_i$ is the ln($k$) change at a cell, $\partial \boldsymbol{h}_j / \partial \Delta \ln(\boldsymbol{k})_i$ is the sensitivity coefficient quantifying the effect on the model response at location $j$ to a change in ln($k$) at location $i$.

The linear optimization routine makes use of the sensitivity matrix, so we compared the coefficients from the two methods on the basis of the accuracy with which they approximate hydraulic head through the linear approximation of Equation (2) around an ensemble of permeability perturbation fields. We hypothesized that *if the Appsen coefficients do as well as the analytically-derived coefficients, they will be adequate for the automatic calibration of parameter fields using linear optimization.*

To make the comparison, 20 alternative unconditional realizations of the permeability field were generated (Figure 7). The difference between the base case permeability vector and each of the 20 realizations was calculated to obtain the Δln(k) perturbation vectors to be applied in Equation (2). These 20 perturbation vectors are meant to mimic proposed perturbations of the permeability field during an optimization routine. The realizations are all sufficiently different to truly test the quality of the sensitivity matrices with respect to approximating the model response by Equation (2). The flow solution was obtained on each realization of the permeability field, using the same boundary conditions and grid discretization, to obtain the 20 vectors of true heads corresponding to the observation locations, denoted $\{h^m\}$. Recall that the Appsen coefficients are returned as simple binary indicators flagging whether a cell has non-zero sensitivity or not, with the appropriate sign attached (Figure 6). To give the Appsen coefficients an appropriate magnitude, a constant value corresponding to the mean of the analytically-calculated sensitivities was applied (i.e., the mean of any of the grid maps shown in Figure 2, which corresponds to a value of approximately 0.03). This is to stress the fact that rather than a precise approximation of the magnitudes of the sensitivity coefficients, it is the correct sign that is critical to the convergence of the parameter optimization. A means of independently coming up with this approximate magnitude and adding a weighting scheme to the values of sensitivity will be added[1].

Figure 8 shows two scatterplots comparing the 20 $\{h^{(lin)}\}$ vectors against the 20 $\{h^{m}\}$ vectors for the 4 observation wells. It is apparent that the Appsen sensitivities perform as well, or perhaps better than the analytically-derived sensitivities, in linearly approximating the heads at observation locations around large parameter field perturbations. In fact, there appears to be less of a bias with the Appsen coefficients. The bias in the analytically-derived sensitivities is attributed to the artifact observed in Figure 2, in which regions outside of the streamtubes of interest are sensitive. Although physically correct, these areas are evidently far less relevant than the cells within the streamtubes containing the observation locations of interest. Further analysis of why this artifact occurs in the analytically-calculated sensitivity coefficients is beyond the scope of this study; it is likely related to the issue of choosing an appropriate scale for a hydrological or reservoir flow problem. In any case, it does not affect the conclusions drawn from the results. The Appsen sensitivities should perform as well as analytically-calculated sensitivity coefficients in inversion routines.

The Appsen method is an extremely fast way to approximate the full sensitivity matrix. The advantage of obtaining the full matrix of sensitivity coefficients is elucidated by Figures 2 and 6. Methods that use parameter optimization must work with a limited subset of parameters as in the master point (Gomez-Hernandez et al., 1997), or pilot point (RamaRao et al., 1995) concept to reduce the degrees of freedom. Optimal choices of the locations to place such master points should be coincident with areas of high sensitivity. High sensitivity regions could be determined with relatively minimal computational effort using the principles outlined in this short note. Practitioners who choose a more traditional trial-and-error approach to model calibration would also benefit from this, saving significant amounts of time in getting to know which areas of the permeability field to focus on. Also, problem areas in the domain which confound the calibration by calling for conflicting parameter adjustments, as a result of data errors for example, can be quickly identified if the full matrix of sensitivities is easily available. These are points which can be developed in future research.

## Conclusions

Sensitivity of pressure response at specific observation locations to the value of grid block permeability at individual model cells is predictable once the pressure solution has been obtained and some means of accounting for the flowpaths, such as particle tracking, has been implemented. The permeability structure coupled with the model specifications, such as boundary conditions and internal sources/sinks (production/injection wells), controls sensitivities. The criteria defining grid block sensitivity with respect to model response at a specific location in

[1] A possible weighting scheme would be to use particle "concentration", such that cells that see more fluid particles in the particle-tracking step of the Appsen algorithm get more weight and thus a greater sensitivity is assigned. Using a weighting scheme to get the approximated elements of the sensitivity matrix would simply improve the accuracy of the linear approximation in Equation (2). However, the fit between the linearly approximated heads and the model-calculated heads does not have to be more precise than what is shown in Figure 8, given the proved performance of the SSC algorithm.

the domain are as follows: (1) cells sharing a group of flowpaths (a streamtube) with the response location under assessment will be the most sensitive cells within the model for that location, (2) bodies of high k cells are sensitive if they are connected to a source or sink in the domain, (3) sensitive regions will coincide with cells of relatively lower permeability that form baffles or obstructions to flow within a streamtube, and (4) the sign of the sensitivity coefficient is related to whether a cell location is up- or downgradient from the observation location within a streamtube. Cells that are downgradient will always have a negative sign, and vice-versa.

These criteria can be used within a standard flow code to output an approximation of the sensitivity matrix, as used in most inversion algorithms, with minimal computational cost. An understanding of the physical controls on the magnitude and sign of sensitivity coefficients used in reservoir model calibration provides the modeler with significant insight into the calibration process. These physical constraints are key to obtaining satisfactory solutions.

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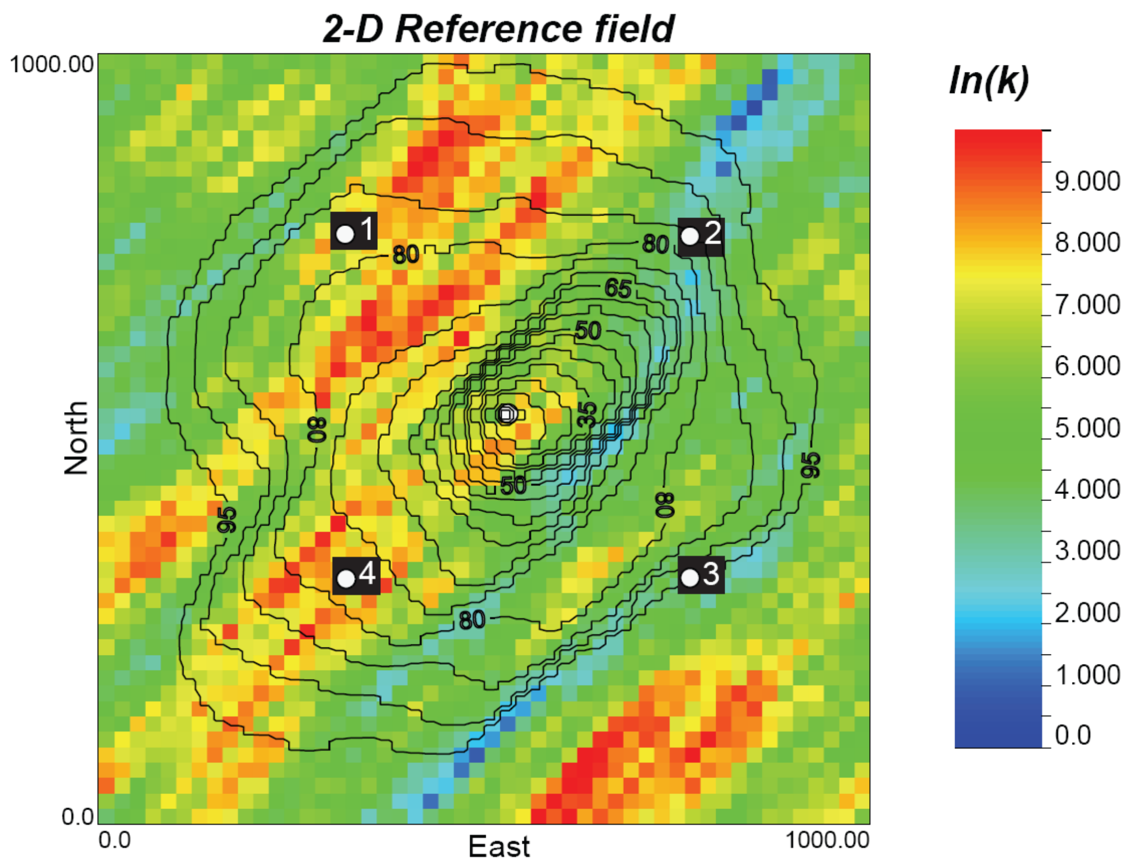


**Figure 1**: Model domain and flow solution. Observation wells are numbered 1-4. Production well at the centre of the domain produces a heterogeneous drawdown pattern indicated by hydraulic head contours, labelled in units of metres.

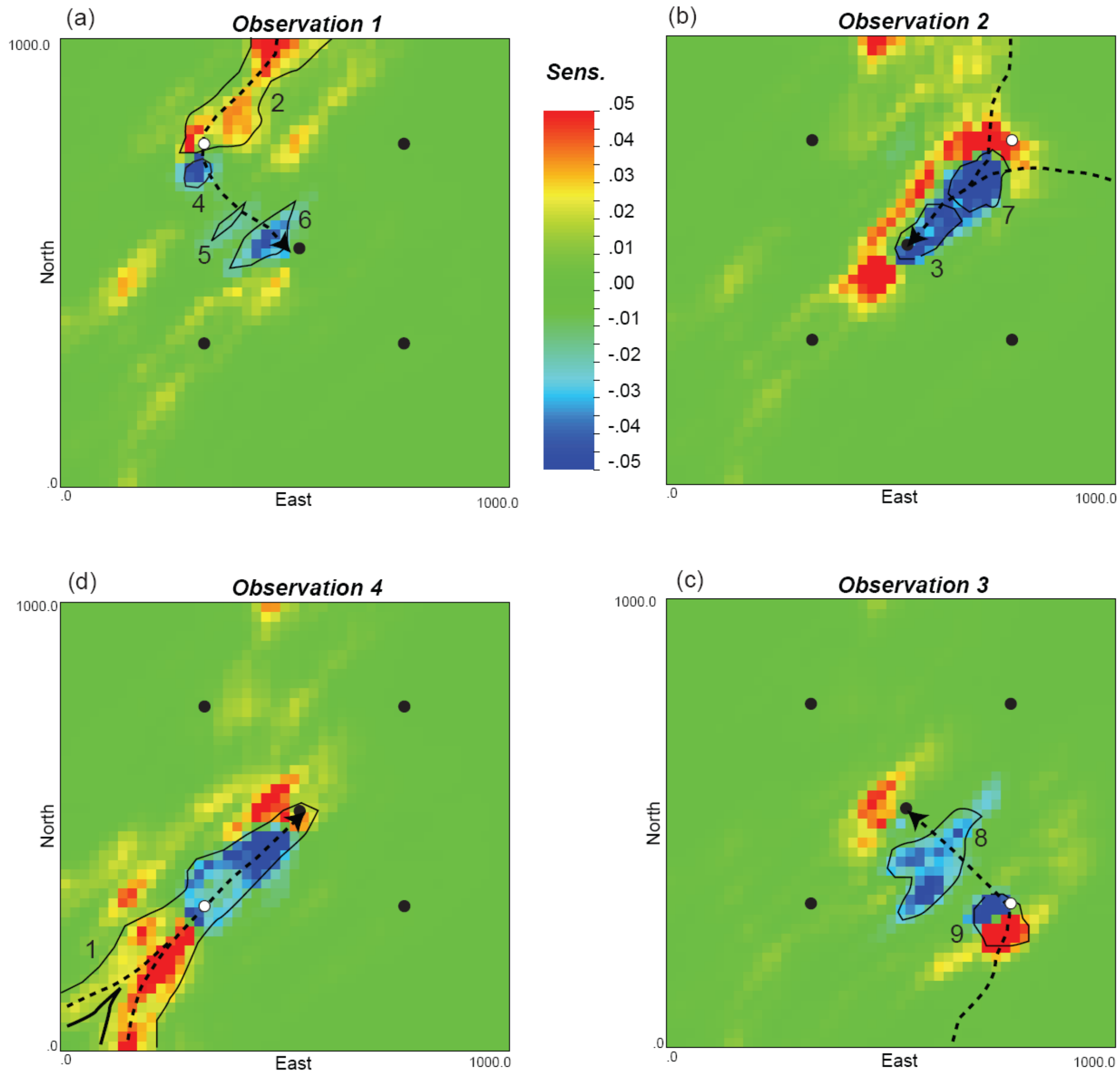


**Figure 2**: Map displays of sensitivity vectors (i.e., rows of the sensitivity matrix) corresponding to the four observation wells. Reds indicate positive sensitivity coefficients and blues are negative sensitivities. Selected bodies of high sensitivity are numbered 1-9 as referenced in the discussion. Dashed arrows indicate flowpaths tracking the centre of streamtubes originating at a boundary, passing through the observation location, and terminating at the production well.

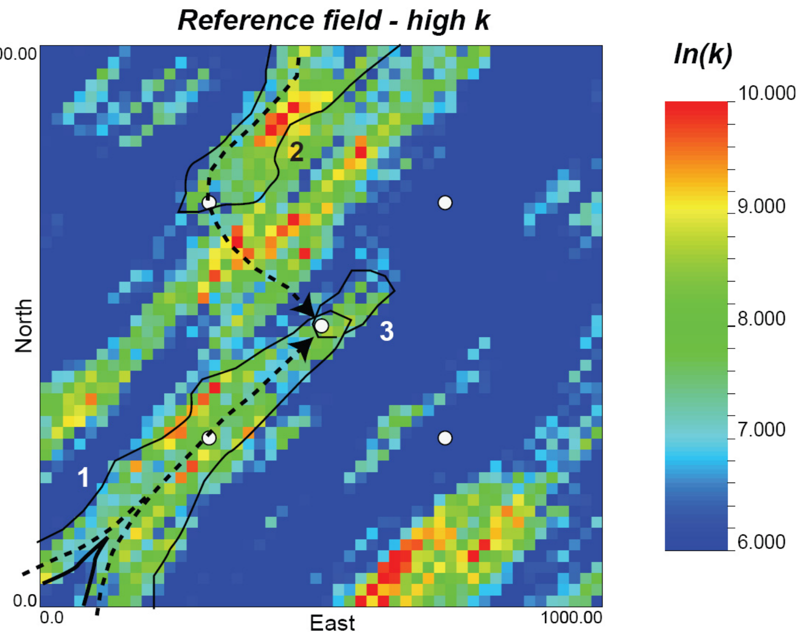


**Figure 3**: Reference permeability field with colour scale highlighting ln(k) bodies which are above the median ln(k) of 6.0. Sensitive bodies numbered 1-3 from Figure 2 correspond to relatively high-k bodies of cells which are connected to a boundary (a boundary condition type that represents a source of fluid) or a well.

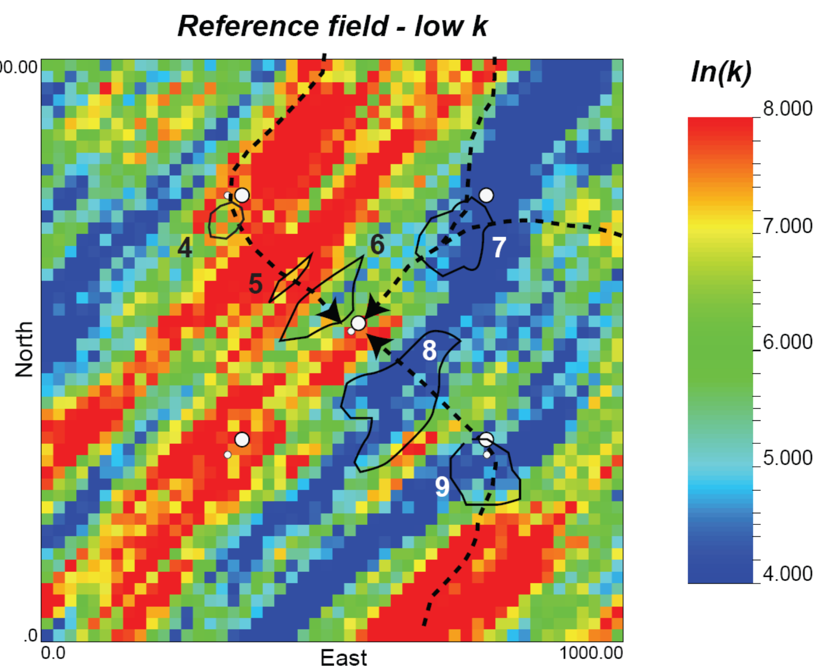


**Figure 4**: Reference permeability field with colour scale range maximizing contrasts in ln(k). Sensitive bodies numbered 4-9 from Figure 2 correspond to relatively low-permeability bodies of cells interrupting otherwise transmissive flowpaths. The permeability contrast may be subtle, and specific to a given streamtube. All that is needed for a body of cells to be sensitive is that it be comprised of relatively low permeability that form a baffle along a flowpath (e.g., bodies 4-9).

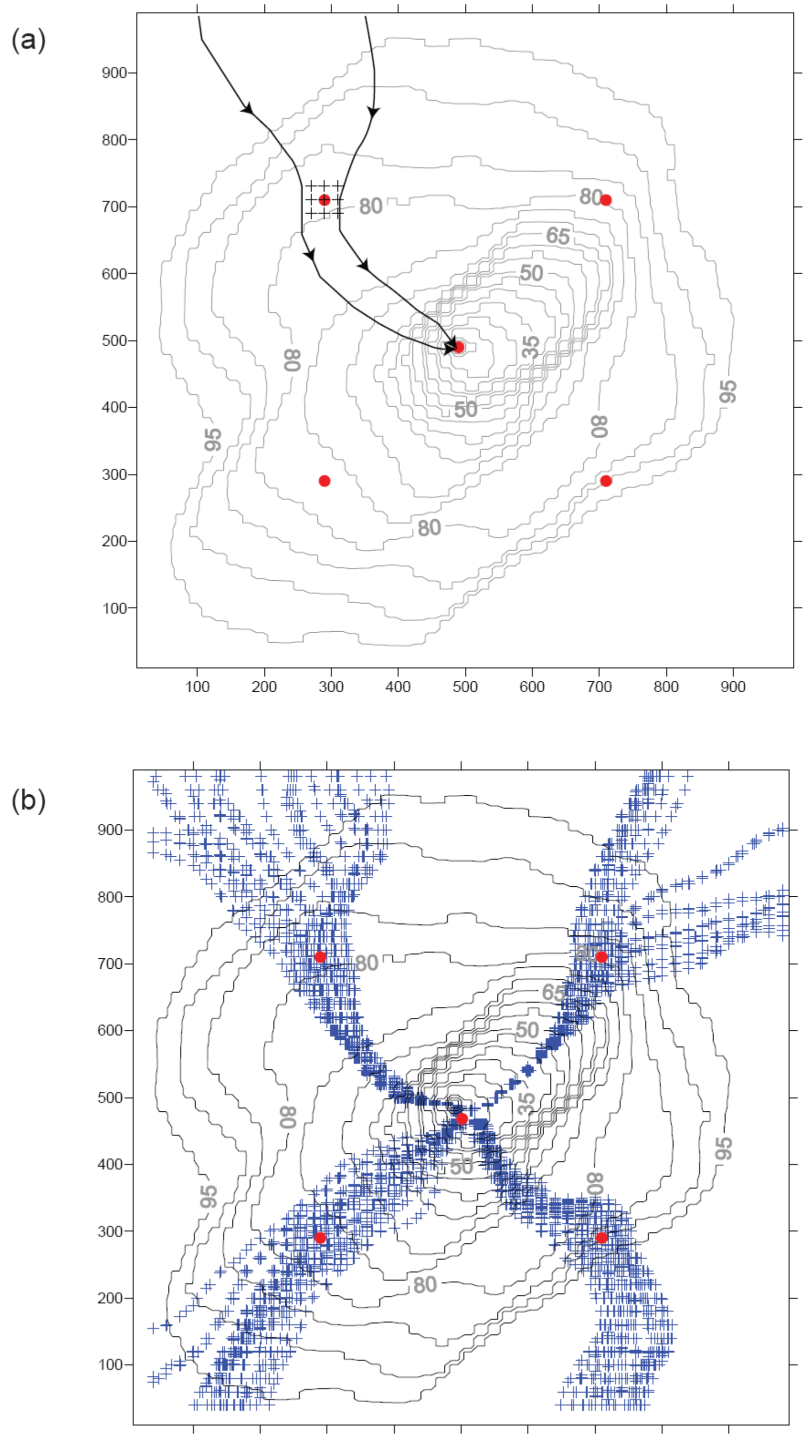


**Figure 5**: (a) Illustration of the definition of an arbitrary streamtube containing observation 1 using a 3x3 particle source template. Particles are tracked forward to the terminus (at the well) and backward to the origin (at the boundary). (b) Streamtubes outlined by particle tracks for the four observation locations.

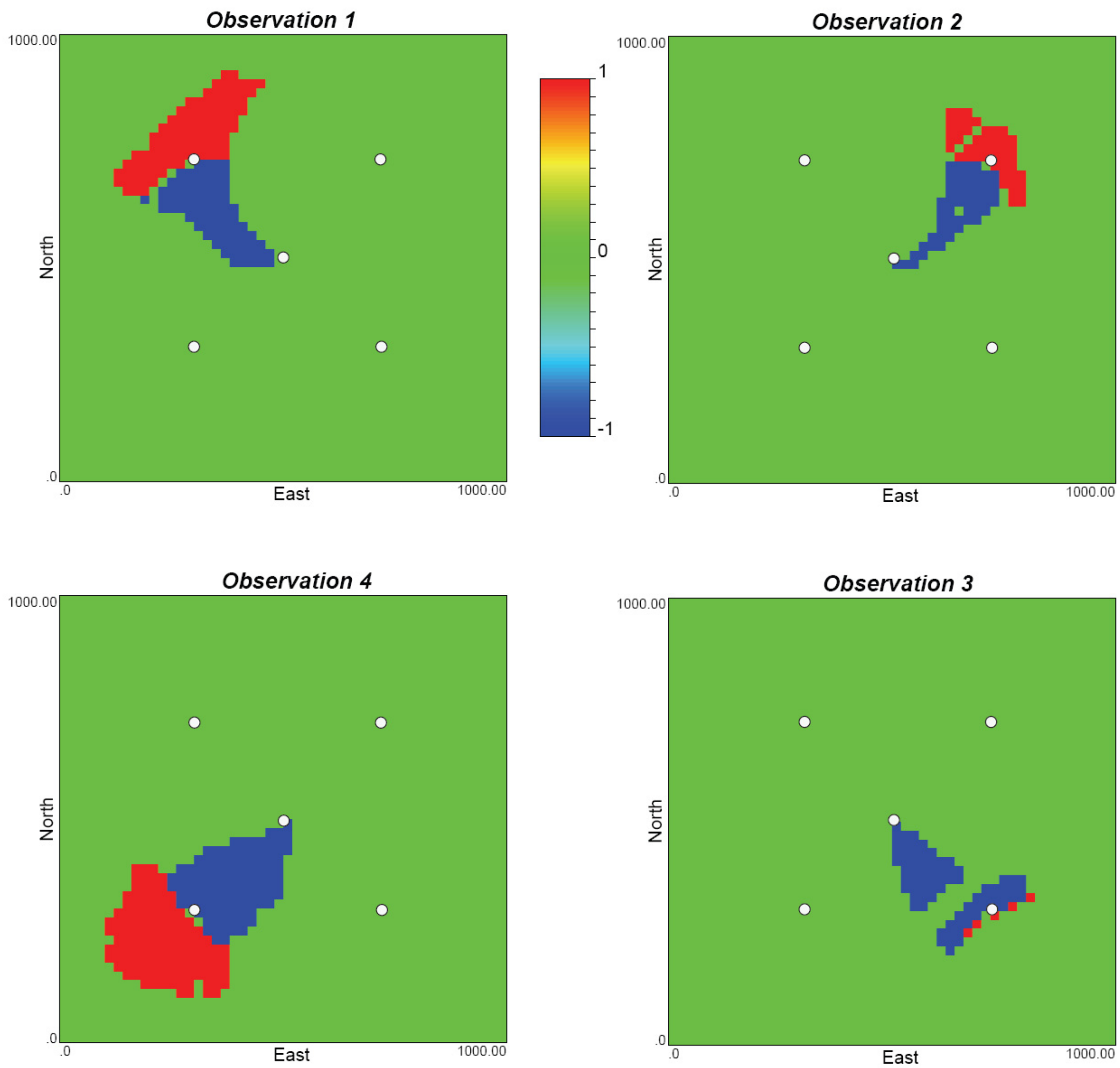


**Figure 6**: Appsen coefficients showing an approximation to the sensitivities observed in Figure 2, but only in terms of cells that have been flagged with a non-zero sensitivity with an appropriate sign.

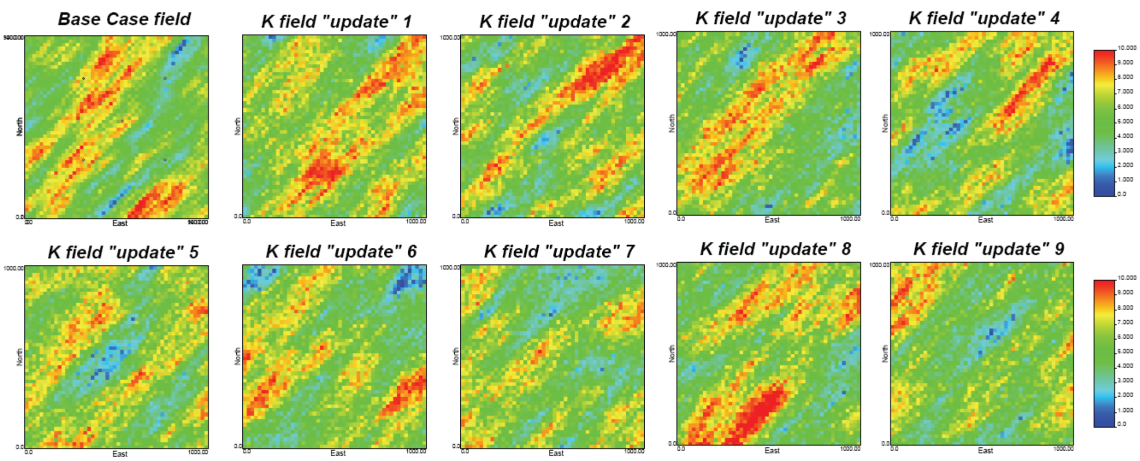


**Figure 7**: 9 of 20 alternative realizations of permeability intended to mimic updates after applying perturbations to the base case field (top left). The difference between the base case permeability vector and any realization is the perturbation vector used in Equation (2) to obtain the linear approximation of head.

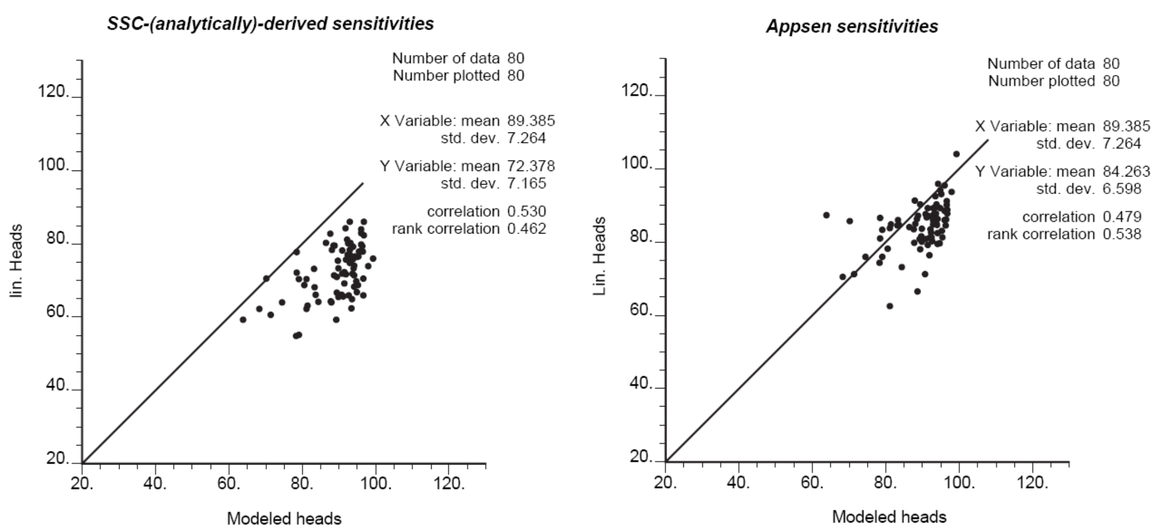


**Figure 8**: (a) Scatterplot comparing the linear approximation of head about 20 permeability perturbation vectors making using the analytically-derived sensitivity coefficients [Equation (2)] against model-calculated heads obtained by solving forward flow model on the 20 resultant permeability fields. (b) The same comparison using the Appsen sensitivities. Note the comparable results in the statistics which are plotted on the diagrams.